ECE 307 – Techniques for Engineering Decisions

Lecture 8b. Dynamic Programming

George Gross

Department of Electrical and Computer Engineering

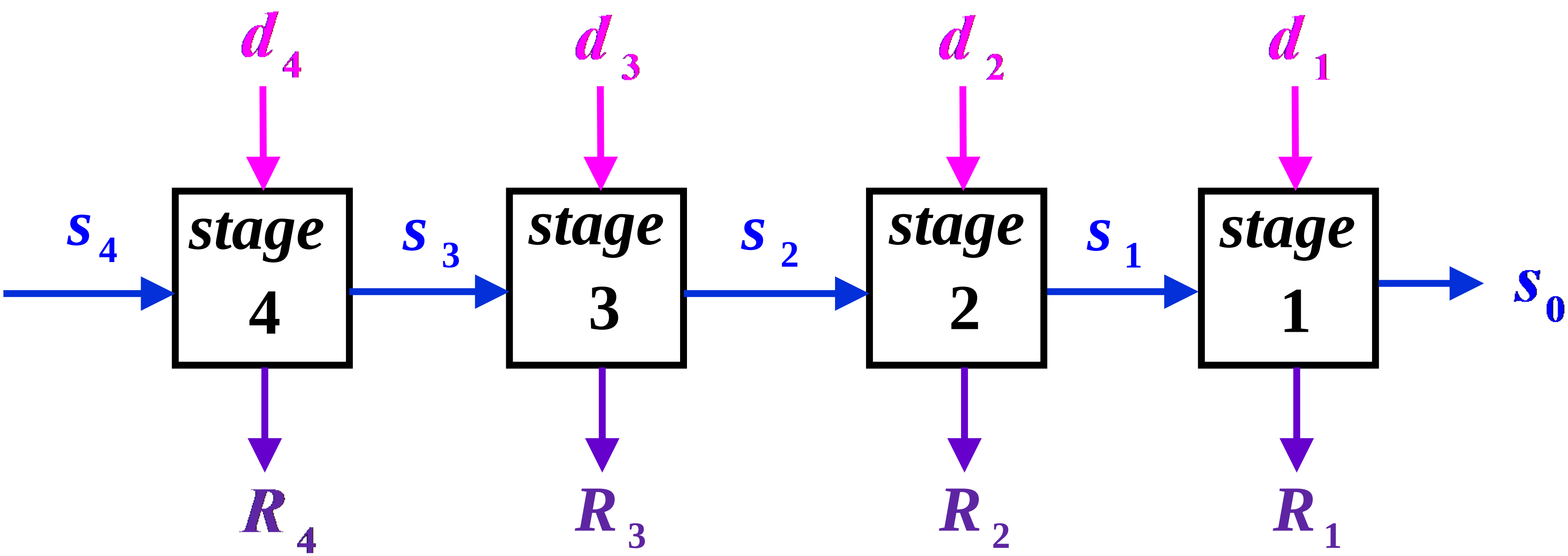
University of Illinois at Urbana-Champaign

OPTIMAL CUTTING STOCK PROBLEM

- A paper company gets an order for:
  - 8 rolls of $2\ ft$ paper sold at $2.50\ \$/roll$
  - 6 rolls of $2.5\ ft$ paper sold at $3.10\ \$/roll$
  - 5 rolls of $4\ ft$ paper sold at $5.25\ \$/roll$
  - 4 rolls of $3\ ft$ paper sold at $4.40\ \$/roll$
- The company only has $13\ ft$ of paper to fill these orders; partial orders may be filled with full rolls
- Determine how to fill orders to maximize

DP SOLUTION APPROACH

- A *stage* is an order and since there are 4 orders we construct a $4 - stage\ \ DP$



DP SOLUTION APPROACH

- A *state* in *stage n* is the remaining *ft* of paper left for the order being processed at *stage n* and all the remaining *stages*
- A decision in *stage n* is the amount of rolls to produce in *stage n* :

DP SOLUTION APPROACH

$$d_n = \left\lfloor \frac{F_0}{L_n} \right\rfloor, \text{ the largest integer in } \frac{F_0}{L_n}$$

where,

L_n = length of order n (ft)

F_0 = length of available paper (ft)

- The *return function* at *stage n* is the additional revenues gained from producing d_n *rolls*

DP SOLUTION APPROACH

- **The *transition function* measures amount of paper remaining at *stage n***

$$s_{n-1} = s_n - d_n L_n \qquad n = 2, 3, 4$$

$$s_0 = s_1 - d_1 L_1$$

and s_0 needs to be as close as possible to *0*

- **Clearly,**

$$d_1 = \left\lfloor \frac{s_1}{L_1} \right\rfloor$$

DP SOLUTION APPROACH

- The recursion relation is

$$f_n^*(s_n) = \max_{0 \leq d_n \leq \left\lfloor \frac{s_n}{L_n} \right\rfloor} \left\{ R_n(s_n, d_n) + f_{n-1}^*(s_{n-1}) \right\}$$

where

$$s_{n-1} = s_n - d_n L_n$$

and

$$f_0^*(s_0) = 0$$

$$f_n(s_n, d_n) = r_n d_n + f_{n-1}^*(s_n - d_n L_n), \quad n = 1, 2, 3, 4$$

DP SOLUTION APPROACH

- **We assume an arbitrary order of the *stages* and pick**

stage n	1	2	3	4
length of order (*ft*)	2.5	4	3	2

- **We proceed backwards from *stage* 1 to *stage* 4 and we know that**

DP SOLUTION: *STAGE* 1

$$f_1^*(s_1) = \max_{0 \le d_1 \le 5} \left\{ r_1(s_1, d_1) \right\} = \max_{0 \le d_1 \le 5} \{ \underbrace{3.10}_{R_1} \, d_1 \}$$

$$d_1 \le \left[\frac{13}{2.5} \right] = 5$$

d_1 \ s_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	-	-	-	3.10	3.10									
2	-	-	-	-	-	6.20	6.20							
3	-	-	-	-	-	-	-	-	9.30	9.30				
4	-	-	-	-	-	-	-	-	-	-	12.40	12.40		
5	-	-	-	-	-	-	-	-	-	-	-	-	-	15.50
$f_1^*(s_1)$	0	0	0	3.10	3.10	6.20	6.20	6.20	9.30	9.30	12.40	12.40	12.40	15.50
d_1^*	0	0	0	1	1	2	2	2	3	3	4	4	4	5

DP SOLUTION: *STAGE* 2

$$f_2^*(s_2) = \max_{0 \le d_2 \le 3} \left\{ 5.25\, d_2 + f_1^*(s_2 - 4\, d_2) \right\}$$

$$d_2 \le \left\lfloor \frac{13}{4} \right\rfloor = 3$$

R_2

d_2 \ s_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	3.10	3.10	6.20	6.20	6.20	9.30	9.30	12.40	12.40	12.40	15.50
1	-	-	-	-	5.25	5.25	5.25	8.35	8.35	11.45	11.45	11.45	14.55	14.55
2	-	-	-	-	-	-	-	-	10.50	10.50	10.50	13.60	13.60	16.70
3	-	-	-	-	-	-	-	-	-	-	-	-	15.75	15.75
$f_2^*(s_2)$	0	0	0	3.10	5.25	6.20	6.20	8.35	10.50	11.45	12.40	13.60	15.75	16.70
d_2^*	0	0	0	0	1	0	0	1	2	1	0	2	3	2

DP SOLUTION: *STAGE* 3

$$f_3^*(s_3) = \max_{0 \le d_3 \le 4} \{ 4.40\, d_3 + f_2^*(s_3 - 3\, d_3) \}$$

$$d_3 \le \left\lfloor \frac{13}{3} \right\rfloor = 4$$

R_3

d_3 \ s_3	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	3.10	5.25	6.20	6.20	8.35	10.50	11.45	12.40	13.60	15.75	16.70
1	-	-	-	4.40	4.40	4.40	7.50	9.65	10.60	10.60	12.75	14.90	15.85	16.80
2	-	-	-	-	-	-	8.80	8.80	8.80	11.90	14.05	15.00	15.00	17.15
3	-	-	-	-	-	-	-	-	-	13.20	13.20	13.20	16.30	18.45
4	-	-	-	-	-	-	-	-	-	-	-	-	17.60	17.60
$f_3^*(s_3)$	0	0	0	4.40	5.25	6.20	8.80	9.65	10.60	13.20	14.05	15.00	17.60	18.45
d_3^*	0	0	0	1	0	0	2	1	1	3	2	2	4	3

DP SOLUTION: *STAGE* 4

$$f_4^*(s_4) = \max_{0 \le d_4 \le 6} \left\{ 2.5\, d_4 + f_3^*(s_4 - 2\, d_4) \right\}$$

$$d_4 \le \left\lfloor \frac{13}{2} \right\rfloor = 6$$

R_4

d_4	*0*	1	2	3	4	5	6	d_4^*	$f_4^*(s_4)$
$s_4 = 13$	18.45	17.5	18.2	17.15	16.2	16.9	15	0	18.45

- The maximum profits are $18.45

DP OPTIMAL SOLUTION

- The *optimal* solution is obtained by retracing

$f_1^*(s_1 = 0) \;=\; 0$ with $d_1^* \;=\; 0 \;\leftrightarrow$ no rolls of $2.5\ ft$

$f_2^*(s_2 = 4) \;=\; 5.25$ with $d_2^* \;=\; 1 \;\leftrightarrow$ 1 roll of $4\ ft$

$f_3^*(s_3 = 13) = 18.45$ with $d_3^* \;=\; 3 \;\leftrightarrow$ 3 rolls of $3\ ft$

$f_4^*(s_4 = 13) = 18.45$ with $d_4^* \;=\; 0 \;\leftrightarrow$ no rolls of $2\ ft$

SENSITIVITY CASE

- **Consider the case that due to an incorrect measurement, in truth, there are only $11\ ft$ available for the rolls**
- **We note that the solution for the original $13\ ft$ covers this possibility in the *stages* $1, 2$ and 3 but we need to re-compute the results of *stage* 4, which we now call *stage* $4'$**

SENSITIVITY CASE : $STAGE\ 4'$

- The *stage* $4'$ computations become

$$d_{4'} \leq \left\lfloor \frac{11}{2} \right\rfloor = 5$$

$d_{4'}$	*0*	1	2	3	4	5	$d^*_{4'}$	$f^*_{4'}(s_4)$
$s_4 = 11$	15	15.7	14.65	13.7	14.4	12.5	1	15.7

- The *optimal* profits in this sensitivity case are $\$15.7$

SENSITIVITY CASE *OPTIMUM*

- The retrace of the solution path obtains

$$d^*_{4'} = 1 \quad \leftrightarrow \quad \text{1 roll of 2 } ft$$

$$d^*_{3'} = 3 \quad \leftrightarrow \quad \text{3 rolls of 3 } ft$$

$$d^*_{2'} = 0 \quad \leftrightarrow \quad 0 \text{ rolls of 4 } ft$$

$$d^*_{1'} = 0 \quad \leftrightarrow \quad 0 \text{ rolls of 2.5 } ft$$

ANOTHER SENSITIVITY CASE

- **We consider the case with the initial 13 *ft*, but in addition we get the constraint that at least 1 roll of 2 *ft* must be produced:**

$$d_4 \geq 1$$

- **Note that no additional work is needed since the computations in the first tables have all the necessary data**
- **This sensitivity case *optimum* profits are *$* 18.2**
- **The *optimum* solution is :**

OPTIMAL CUTTING STOCK PROBLEM

$f^*_{4''}(s_4 = 13) \;=\; 18.2$ **with** $d^*_{4''} = 2 \;\leftrightarrow\;$ **2 rolls of 2** ft

$f^*_{3''}(s_3 = 9) \;=\; 13.2$ **with** $d^*_{3''} = 3 \;\leftrightarrow\;$ **3 rolls of 3** ft

and since $s_2 = s_1 = 0$ $\quad d^*_{2''} = 0 \;\leftrightarrow\;$ ***0*** **rolls of 4** ft

$d^*_{1''} = 0 \;\leftrightarrow\;$ ***0*** **rolls of 2.5** ft

- **The additional constraint reduces the *optimum* from *$* 18.45 to *$*18.2 and so it costs *$* .25**

INVENTORY CONTROL PROBLEM

- This problem is concerned with the development of an *optimal* ordering policy for a retailer
- The sales of a seasonal item has the demands

month	*Oct*	*Nov*	*Dec*	*Jan*	*Feb*	*Mar*
demand	40	20	30	40	30	20

INVENTORY CONTROL PROBLEM

- **All units sold are purchased from a vendor at 4 *$/unit* ; units are sold in lots of 10, 20, 30, 40 or 50 with the corresponding discount**

lot size	10	20	30	40	50
discount %	4	5	10	20	25

INVENTORY CONTROL PROBLEM

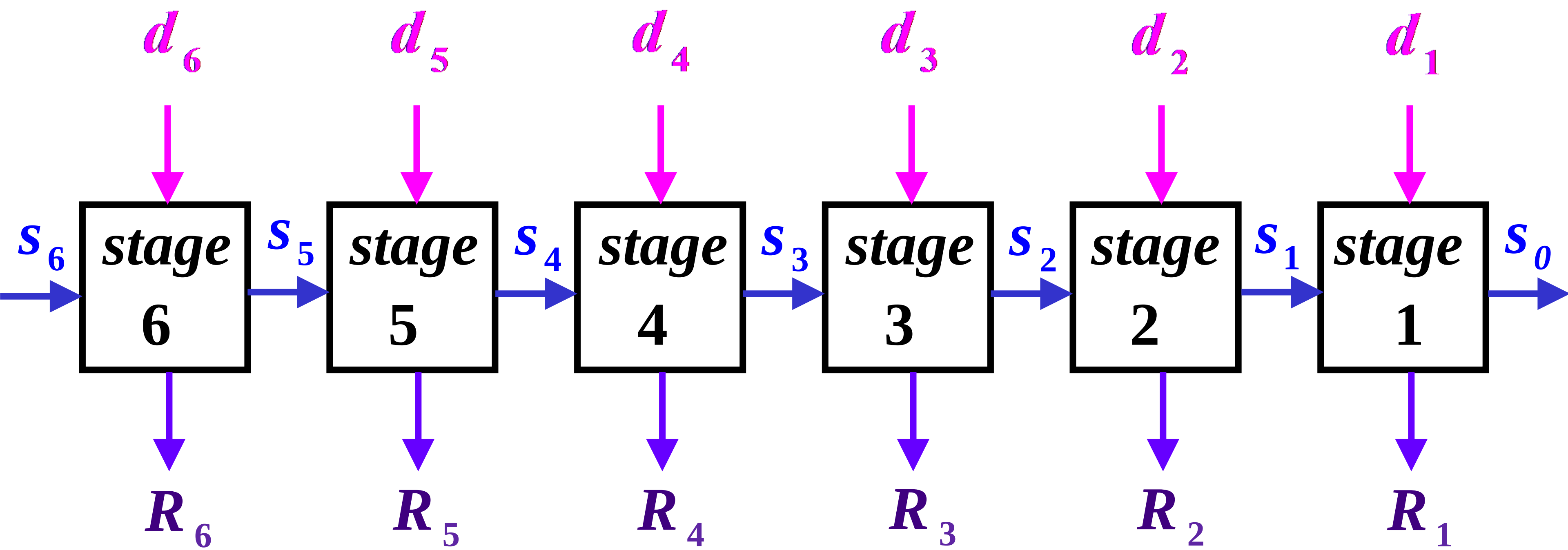
- There are additional ordering costs: each order incurs fixed costs of $\$\ 2$ and $\$\ 8$ for shipping, handling and insurance
- The storage limitations of the retailer require that no more than 40 units be in inventory at the end of the month and the storage charges are $0.2\ \$/unit$; there is 0 inventory at the beginning and at the end of the period under consideration
- Underlying assumption: demand occurs at a constant rate throughout each month

DP SOLUTION APPROACH

- **We formulate the problem as a *DP* and use a backward process for solution**
- **Each *stage* corresponds to a month**

month	*Oct*	*Nov*	*Dec*	*Jan*	*Feb*	*Mar*
stage n	6	5	4	3	2	1

DP SOLUTION APPROACH



- R_n is the contribution to the total cost of the ordering policy from the stage n decision, $n = 1, 2, \ldots, 6$

DP SOLUTION APPROACH

- **The *state* variable s_n in *stage n* is defined as the amount of inventory that is stored from the previous month, taking into account that n additional months remain in the planning period – the month corresponding to stage n plus the months in the stages $n - 1$, $n - 2$, ... , 1**

DP SOLUTION APPROACH

- The decision variable d_n in *stage n* is the amount of units ordered to satisfy the n remaining months' demands D_n and D_i , $i = n-1, n-2, \ldots, 2, 1$

- The transition function is defined by

$$s_{n-1} = s_n + d_n - D_n \qquad n = 1, 2, \ldots, 6$$

$$s_0 = 0 \qquad s_6 = 0$$

demand in month n

DP SOLUTION APPROACH

- The *return function* in the *stage* n is given by

$$r_1(s_1, d_1) = \underbrace{\phi(d_n)}_{\substack{\textit{ordering} \\ \textit{costs}}} + \underbrace{h_n(s_n + d_n - D_n)}_{\substack{0.2(s_n + d_n - D_n) \\ \textit{storage costs}}}$$

with

$$d_n = 0, 10, 20, 30, 40 \text{ or } 50$$

$$\phi(d_n) = \underbrace{10}_{\substack{\textit{fixed} \\ \textit{costs}}} + 4[1 - \underbrace{\rho(d_n)}_{\substack{\textit{discount} \\ \textit{factor}}}]\, d_n \text{ for } d_n = 10, 20, 30, 40, 50$$

$$\phi(d_n) = 0 \quad \text{for} \quad d_n = 0$$

DP SOLUTION APPROACH

d_n	0	10	20	30	40	50
$\phi(d_n)$	0	48	86	118	138	160

- In the *DP* approach, at each *stage n*, we minimize the costs for the order in the stage $n, n-1, \ldots, 1$

$$f_n^*(s_n) = \min_{d_n}\left\{ \phi(d_n) + h_n\left[s_n + d_n - D_n\right] + f_{n-1}^*(s_{n-1}) \right\}$$

$$n = 1, \ldots, 6$$

$f(s_0) = 0$ and so $f_0^*(s_0) = 0$

DP SOLUTION: *STAGE* 1

$$\left.\begin{array}{l} s_0 = 0 \\ D_1 = 20 \end{array}\right\} \Rightarrow s_1 = 20, 10 \text{ or } 0 \Rightarrow d_1^* = 0, 10 \text{ or } 20$$

$$f_1^*(s_1) = \min_{d_1} \left\{ \phi(d_1) + 0 \right\} = \phi(d_1^*)$$

s_1	20	10	0
d_1^*	0	10	20
$f_1^*(s_1)$	0	48	86

DP SOLUTION: *STAGE* 2

$$s_1 = s_2 + d_2 - 30 \text{ since } D_2 = 30$$

$$f_2^*(s_2) = \min_{d_2} \left\{ \phi(d_2) + 0.2\left[s_2 + d_2 - 30 \right] + f_1^*(s_1) \right\}$$

s_2	d_2						d_2^*	$f_2^*(s_2)$
	0	10	20	30	40	50		
0				204	188	164	50	164
10			172	168	142		40	142
20		134	136	122			30	122
30	86	98	90				*0*	86
40	50	52					*0*	50

DP SOLUTION: *STAGE* 3

$$s_2 = s_3 + d_3 - 40 \text{ since } D_3 = 40$$

$$f_3^*(s_3) = \min_{d_3}\left\{ \phi(d_3) + 0.2\Big[\underbrace{s_3 + d_3 - 40}_{s_2}\Big] + f_2^*(s_2) \right\}$$

s_3	d_3						d_3^*	$f_3^*(s_3)$
	0	10	20	30	40	50		
0					302	304	40	302
10				282	282	286	30, 40	282
20			250	262	264	252	20	250
30		212	230	244	230	218	10	212
40	164	192	212	210	196		*0*	164

DP SOLUTION: *STAGE* 4

$s_3 = s_4 + d_4 - 30$ since $D_4 = 30$

$$f_4^*(s_4) = \min_{d_4}\left\{ \phi(d_4) + 0.2\underbrace{\left[s_4 + d_4 - 30\right]}_{s_3} + f_3^*(s_3) \right\}$$

s_4	d_4						d_4^*	$f_4^*(s_4)$
	0	10	20	30	40	50		
0				420	422	414	50	414
10			388	402	392	384	50	384
20		350	370	372	362	332	50	332
30	302	332	340	342	310		*0*	302
40	284	302	310	290			*0*	284

DP SOLUTION: *STAGE* 5

$$s_4 = s_5 + d_5 - 20 \text{ since } D_5 = 20$$

$$f_5^*(s_5) = \min_{d_5}\left\{ \phi(d_5) + 0.2\left[\underbrace{s_5 + d_5 - 20}_{s_4}\right] + f_5^*(s_5) \right\}$$

s_5	d_5						d_5^*	$f_5^*(s_5)$
	0	10	20	30	40	50		
0			500	504	474	468	50	468
10		462	472	454	446	452	40	446
20	414	434	422	426	430		*0*	414
30	386	384	394	410			10	384
40	336	356	378				*0*	336

DP SOLUTION: *STAGE* 6

$D_6 = 40$ and $s_6 = 0$

$s_5 = s_6 + d_6 - 40 = d_6 - 40$

$$f_6^*(s_6) = \min_{d_6}\left\{ \phi(d_6) + 0.2\Big[\underbrace{s_6 + d_6 - 40}_{s_5}\Big] + f_5^*(s_5) \right\}$$

d_6	*0*	10	20	30	40	50	d_6^*	$f_6^*(s_6)$
$f_6(s_6)$					606	608	40	606

$d_6^* = 40 \Rightarrow d_5^* = 50 \Rightarrow d_4^* = 0 \Rightarrow d_3^* = 40 \Rightarrow d_2^* = 50 \Rightarrow d_1^* = 0$

OPTIMAL SOLUTION

$d_6^* = 40$ which implies $s_5 = 0$ and costs 606

$d_5^* = 50$ which implies $s_4 = 30$ and costs 468

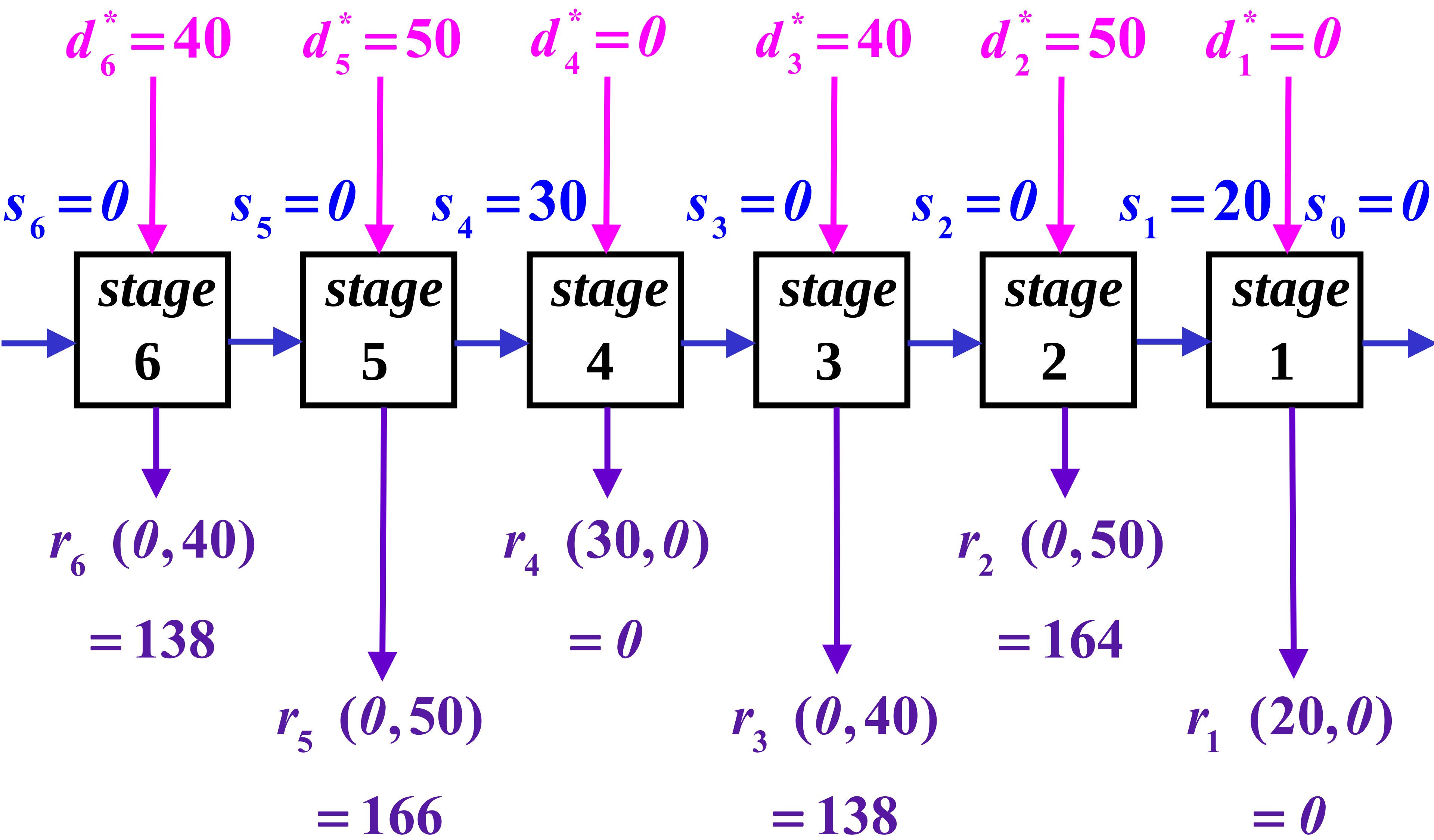
$d_4^* = 0$ which implies $s_3 = 0$ and costs 302

$d_3^* = 40$ which implies $s_2 = 0$ and costs 302

$d_2^* = 50$ which implies $s_1 = 20$ and costs 164

$d_1^* = 0$ with costs *0*

OPTIMAL SOLUTION



OPTIMAL SOLUTION

- The *optimal* trajectory is

$$s_0 = 0 \longrightarrow s_1 = 20 \longrightarrow s_2 = 0 \longrightarrow s_3 = 0 \longrightarrow s_4 = 30 \longrightarrow s_5 = 0$$

- The total costs for the sequence of orders are given by

$$0 + 164 + 138 + 0 + 166 + 138 = 606$$

MUTUAL FUND INVESTMENT STRATEGIES

- We consider a 5–year investment of
  - 10 *k$* invested in year 1
  - 1 *k$* invested in each year 2, 3, 4 and 5 into 2 mutual funds with different yields for both the short-term (1 year) and the long-term (up to 5 years)
- The decision on the allocation of investment in each fund is made at the beginning of each year

MUTUAL FUND INVESTMENT STRATEGIES

- We operate under the following protocol:
  - each fund returns short–term dividends and long–term dividends
  - once invested, the money cannot be withdrawn until the end of the 5 – year period
  - all short–term gains may either be reinvested in one of the two funds, or withdrawn; in the latter case, the withdrawn funds earn no further interest
- Our objective is to maximize the total returns at the end of 5 years

MUTUAL FUND INVESTMENT STRATEGIES

- **The earnings on the investment are**
  - ***LTD* : the long–term dividend specified as *% / year* return on the accumulated capital**
  - ***STD* : the short–term interest dividend returned as cash to the investor at the end of the period; cash may be invested in either fund and any money not invested earns no return**

MUTUAL FUND INVESTMENT PARAMETERS

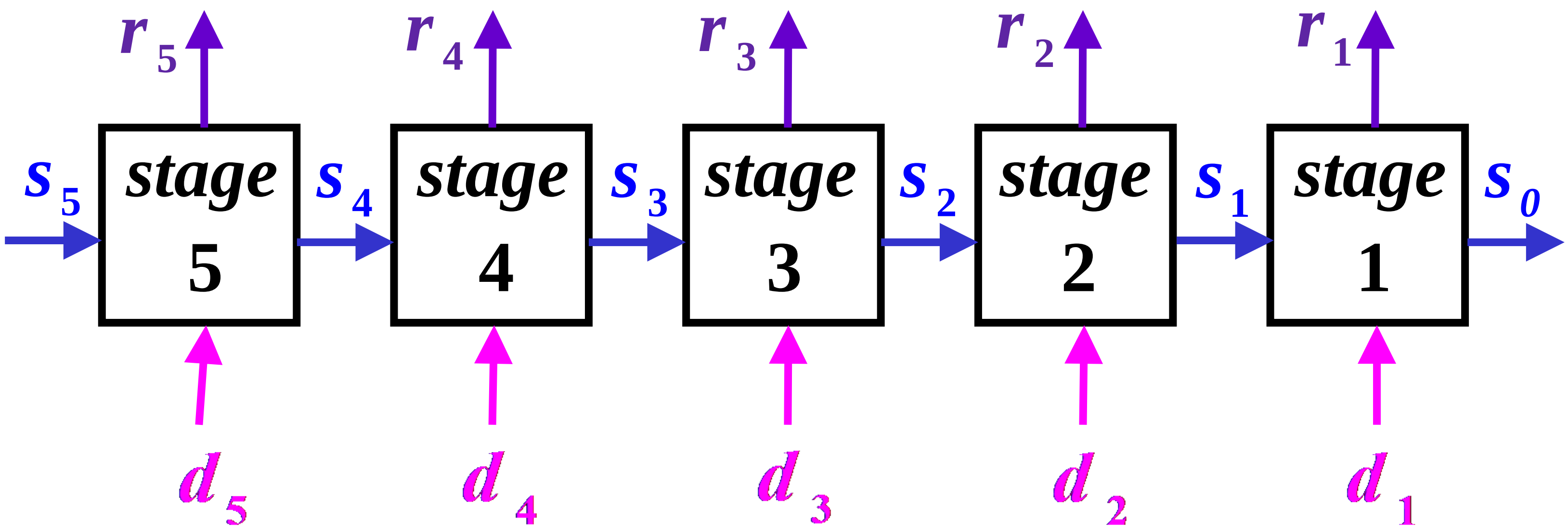
fund	*STD rate i_n for year n*					*LTD rate I*
	1	2	3	4	5	
A	0.02	0.0225	0.0225	0.025	0.025	0.04
B	0.06	0.0475	0.05	0.04	0.04	0.03

DP SOLUTION APPROACH

- We use backwards *DP* to solve the problem

- The *stages* are the 5 investment periods

$$stage\ n \triangleq year\ 6-n \qquad n = 1,2,3,4,5$$



DP SOLUTION METHOD

- For *stage* n , the *state* s_n is the capital available for investment in the year $6-n$
- The decision d_n is the amount of capital invested in fund A in year $6-n$ and so the amount of capital invested in fund B in the year $6-n$ is $s_n - d_n$
- In each year, we determine the amount to invest in fund A and in fund B in order to optimize the returns at the end of year 5

DP SOLUTION METHOD

- The *backward recursion* application considers year 5 first and then each previous year in sequence
- Basic considerations:
  - for each of the stages $6 - n$, $n = 1, \ldots, 5$, d_n is invested in fund A with returns $d_n i_A (STD)$ and $(s_n - d_n)$ is invested in fund B with returns $(s_n - d_n) i_B (STD)$

DP SOLUTION METHOD

- for the stage $6 - n + 1$, the *STD*s are augmented by \$1,000

$$s_{n-1} = d_n i_A + (s_n - d_n) i_B + 1,000 \qquad n = 2,3,4,5$$

- For the stage 5, we have the initial investment

$$s_5 = 10,000$$

THE OBJECTIVE

- The objective is to maximize the total returns

$$max\ R = \sum_{n=1}^{5} r_n \quad \text{evaluated at the end of year } 5$$

- We express *all* returns in the end of the year 5 dollars: r_n is the future value of long –term earnings in the years $1, 2, 3$ and 4

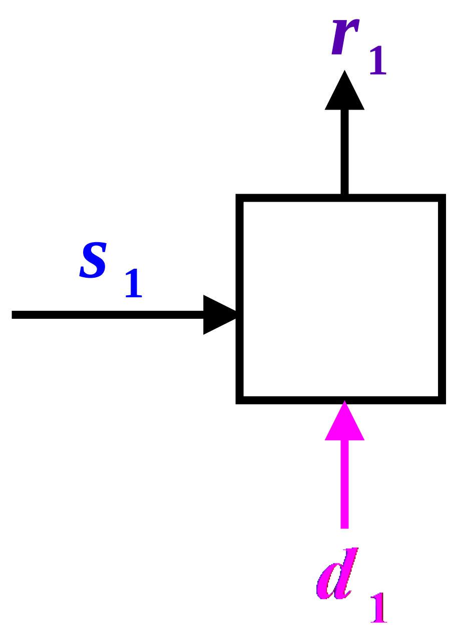
$$r_n = (1 + I_A)^n d_n + (1 + I_B)^n (s_n - d_n) \qquad n = 1, \ldots, 5$$

- But for $n = 1$, r_1 is the present value of all earnings in *stage* 1

$$r_1 = (1 + I_A)\, d_1 + (1 + I_B)(s_1 - d_1) + i_A d_1 + i_B (s_1 - d_1)$$

DP SOLUTION: *STAGE* 1

- For *stage* 1



where

$$r_1 = (1 + I_A) d_1 + (1 + I_B)(s_1 - d_1) + i_{1A} d_1 + i_{1B}(s_1 - d_1)$$

$$= \left(I_A + i_{1A} - I_B - i_{1B} \right) d_1 + (1 + I_B + i_{1B}) \, s_1$$

DP SOLUTION: *STAGE* 1

- r_1 = earnings in *stage* 1 (associated with the *stage* 1 decision)

$$f_1^*(s_1) = \max_{d_1} \{r_1\} = \max_{d_1} \left\{ \begin{array}{l} d_1(I_A + i_{1A} - I_B - i_{1B}) + \\ s_1(1 + I_B + i_{1B}) \end{array} \right\}$$

$$= \max_{0 \le d_1 \le s_1} \left\{ \begin{array}{l} d_1(0.04 + 0.025 - 0.03 - 0.04) + \\ s_1(1 + 0.03 + 0.04) \end{array} \right\}$$

$$= \max_{d_1} \{ d_1(-0.005) + s_1(1.07) \}$$

optimal decision

$$d_1^* = 0 \quad \text{with} \quad f_1^*(s_1) = 1.07 s_1$$

maximum return in *stage* 1

DP SOLUTION: *STAGE* 2

- r_2 = returns associated with the decision in stage 2 realized at the end of 5 years

$$= d_2 (1 + I_A)^2 + (s_2 - d_2)(1 + I_B)^2$$

$$= d_2 \left[(1 + I_A)^2 - (1 + I_B)^2 \right] + s_2 (1 + I_B)^2$$

- As a consequence of the decision d_2, the funds for investment in stage 1 are

$$s_1 = s_2 i_{1B} + d_2 (i_{1A} - i_{1B}) + 1,000$$

DP SOLUTION: *STAGE* 2

- We select d_2^* to maximize

$$f_2^*(s_2) = \max_{d_2} \left\{ r_2 + f_1^*(s_1) \right\}$$

$$= \max_{0 \le d_2 \le s_2} \left\{ \begin{array}{l} d_2(.0207) + 1.0609 s_2 + \\ 1.07[\,.04 s_2 + d_2(-.015) + 1{,}000] \end{array} \right\}$$

$$= \max_{d_2} \left\{ d_2 (1.04^2 - 1.03^2) + s_2 (1.03)^2 + f_1^*(s_1) \right\}$$

$$= \max_{d_2} \left\{ d_2 (.0046) + 1.1037 s_2 + 1{,}070 \right\}$$

$d_2^* = s_2$ with $f_2^*(s_2) = 1.108 s_2 + 1{,}070$

DP SOLUTION: *STAGE* 3

- r_3 = **returns associated with the decision d_3 realized at the end of 5 years**

$$= d_3 (1 + I_A)^3 + (s_3 - d_3)(1 + I_B)^3$$

$$= d_3 \left[(1 + I_A)^3 - (1 + I_B)^3 \right] + s_3 (1 + I_B)^3$$

- **As a consequence of the decision d_3, the funds for investment in stage 2 are**

$$s_2 = s_3 \, i_{3B} + d_3 (i_{3A} - i_{3B}) + 1,000$$

DP SOLUTION: *STAGE* 3

- We select d_3^* to maximize

$$f_3^*(s_3) = \max_{d_3} \left\{ r_3 + f_2^*(s_2) \right\}$$

$$= \max_{d_3} \left\{ \begin{array}{l} d_3 (1.04^3 - 1.03^3) + s_3 (1.03)^3 + \\ 1.108 s_2 + 1,070 \end{array} \right\}$$

$$= \max_{0 \leq d_3 \leq s_3} \left\{ 2,178 + 1.1481 s_3 + .0018 d_3 \right\}$$

$$d_3^* = s_3 \quad \text{with} \quad f_3^*(s_3) = 1.15 s_3 + 2,178$$

DP SOLUTION: *STAGE* 4

- r_4 = returns associated with the decision d_4 realized at the end of 5 years

$$= d_4 (1 + I_A)^4 + (s_4 - d_4)(1 + I_B)^4$$

$$= d_4 \left[(1 + I_A)^4 - (1 + I_B)^4 \right] + s_4 (1 + I_B)^4$$

- The funds for investment in stage 3 depend explicitly on d_4

$$s_3 = s_4 \, i_{4B} + d_4 (i_{4A} - i_{4B}) + 1{,}000$$

DP SOLUTION: *STAGE* 4

- We select d_4^* to maximize

$$f_4^*(s_4) = \max_{d_4} \left\{ r_4 + f_3^*(s_3) \right\}$$

$$= \max_{d_4} \left\{ d_4 (1.04^4 - 1.03^4) + s_4 (1.03)^4 + 1.15 s_3 + 2{,}178 \right\}$$

$$= \max_{0 \le d_4 \le s_4} \left\{ 3328 + 1.1772\, s_4 + .0156\, d_4 \right\}$$

$$d_4^* = s_4 \quad \text{with} \quad f_4^*(s_4) = 1.193\, s_4 + 3{,}328$$

DP SOLUTION: *STAGE* 5

- r_5 = returns associated with the decision d_5 realized at the end of 5 years

$$= d_5 (1 + I_A)^5 + (s_5 - d_5)(1 + I_B)^5$$

$$= d_5 \left[1.04^5 - 1.03^5 \right] + s_5 \, (1.03)^5$$

- The funds available in stage 5 are

$$s_5 = 10,000$$

- Therefore, the funds available for investment in stage 4 are

DP SOLUTION: *STAGE* 5

$$s_4 = s_5\, i_{5B} + d_5(i_{5A} - i_{5B}) + 1,000$$

$$= 10,000\, i_{5B} + d_5(i_{5A} - i_{5B}) + 1,000$$

- We select d_5^* to maximize

$$f_5^*(s_5) = \max_{0 \le d_5 \le s_4} \left\{ \underbrace{10,000(1.03)^5}_{11,593} + d_5 \underbrace{(1.04^5 - 1.03^5)}_{0.0574} + f_4^*(s_4) \right\}$$

$$\left[\underbrace{1,000 + 600 + d_5(-.04)}_{s_4} \right] 1.193 + 3,328$$

DP SOLUTION: *STAGE* 5

$$= \max_{0 \leq d_5 \leq s_5} \left\{ 16{,}830 + d_5 \underbrace{(0.0574 - 0.048)}_{0.097} \right\}$$

$$= 16{,}830 + 0.097(10{,}000)$$

$d_5^* = 10{,}000$ **with** $f_5^*(s_5) = 16{,}927$

OPTIMAL SOLUTION

optimal **return at end of 5 years is 16,927 using the following strategy**

beginning of year	*investment in*	
	fund A	*fund B*
1	**10,000**	***0***
2	***STD returns* + 1,000**	***0***
3	***STD returns* + 1,000**	***0***
4	***STD returns* + 1,000**	***0***
5	***0***	***STD returns* + 1,000**